

Real and pseudoreal forms of D=4 complex Euclidean (super)algebras and super-Poincaré/ super-Euclidean r-matrices

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Abstract. We provide the classification of real forms of complex D=4 Euclidean algebra $\mathfrak{e}(4; \mathbb{C}) = \mathfrak{o}(4; \mathbb{C}) \ltimes \mathbf{T}_{\mathbb{C}}^4$ as well as (pseudo)real forms of complex D=4 Euclidean superalgebras $\mathfrak{e}(4|N; \mathbb{C})$ for $N=1, 2$. Further we present our results: $N=1$ and $N=2$ supersymmetric D=4 Poincaré and Euclidean r-matrices obtained by using D= 4 Poincaré r-matrices provided by Zakrzewski [1] For $N=2$ we shall consider the general superalgebras with two central charges.

1. Introduction

The classification of quantum deformations of Lorentz symmetries described by classical r-matrices was given firstly by Zakrzewski [2] (see also [3]), and has been further extended to the classification of classical r-matrices for Poincaré algebra in [1]. The classification of dual Hopf-algebraic quantum deformations of Poincaré group were presented in [4]. Subsequently, the infinitesimal r-matrix description of the deformations of Poincaré algebra presented in [1] has been extended in several papers to finite Hopf-algebraic deformations, with conclusive results obtained in [5]. Because majority of studied deformations were triangular, i.e. described by the twist deformations, they permitted (see e.g. [6, 7]) to derive explicit formulas for the non-commutative algebra of deformed space-time coordinates by the use of so-called star product realizations.

The study of deformations of spacetime supersymmetries and the corresponding deformed superspaces were less systematic, related mostly either with the supersymmetrization of simplest Abelian canonical twist deformation of Poincaré symmetries [8]–[10] or with the supersymmetric extension of κ -deformation [11, 12]. The supersymmetrization of such a canonical twist F for $N = 1$ Poincaré superalgebra looks as follows

$$F = \exp \frac{1}{2} \theta^{\mu\nu} P_{\mu} \wedge P_{\nu} \rightarrow \mathcal{F} = \exp \frac{1}{2} \theta^{\mu\nu} P_{\mu} \wedge P_{\nu} \exp \xi^{\alpha\beta} Q_{\alpha} \wedge Q_{\beta}, \quad (1.1)$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$, $\xi^{\alpha\beta} = \xi^{\beta\alpha}$ and $\hat{x} \wedge \hat{y} \equiv (\hat{x} \otimes \hat{y}) - (-1)^{|x||y|} (\hat{y} \otimes \hat{x})$ where $|x| = 0, 1$ is the Z_2 -grading of the superalgebra element \hat{x} with definite grading. The classification of the supersymmetric triangular deformations for finite-dimensional simple Lie superalgebras was firstly given in a mathematical framework by one of the present authors [13]; one should add also that already long time ago the

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nontriangular Drinfeld-Jimbo deformation has been provided for all complex finite-dimensional simple Lie superalgebras in [14, 15, 16] but without discussion of its real forms. Recently however (see [17, 19]) the classification of Poincaré r -matrices in [2] has been extended to $D = 4, N = 1$ Poincaré and Euclidean [19] supersymmetries, which provided physically important deformations of non-semisimple Lie superalgebras. New result in this paper is the presentation of the list of $D=4$ $N=2$ classical super-Poincaré and super-Euclidean r -matrices.

We add that there were also considered twisted Euclidean $N = 1$ and $N = 2$ supersymmetries (see e.g. [9]) but only under the assumption that the fermionic part of the twist factor was a supersymmetric enlargement of canonical Abelian twist (see (1.1)). In our recent paper by considering partial classification of $N = 1$ complex r -matrices and their pseudoreal (Euclidean) and real (Poincaré) forms [19] we have obtained large class of $N = 1$ supersymmetric twists which provide new $D=4$ superspaces with Lie-algebraic deformation of bosonic spacetime sector.

In quantum deformations approach the basic primary notion is the Hopf-algebraic deformation of (super)symmetries which subsequently implies the modification of (super)spacetime algebra. It follows that for classifying the possible deformations of (super)symmetries one should list the deformations of corresponding Hopf (super)algebras. In non-supersymmetric case the most complete discussion of such Hopf-algebraic approach to field theories deformed by canonical twists F (see (1.1)) was presented in [18]; for Poincaré and Euclidean supersymmetries such way of introducing deformations via supertwist, firstly advocated in [8, 9], was presented recently in [19, 20]. In present paper we supplement classical $N = 2$ super-Poincaré r -matrices which generate the corresponding $N=2$ supersymmetric twist (triangular) deformations. Subsequently using \ast -product formulation [6, 7] one can provide effective formulae for the description of quantum-deformed $N=2$ superspaces. We add that the particular supersymmetric $N = 2$ twist deformations considered in earlier studies [21]–[25] appear as particular cases in the list of $N=2$ deformations presented in this paper.

In Sect. 2 and 3 we shall consider in some detail inhomogeneous $\mathfrak{o}(4; \mathbb{C})$ algebra with its real forms and the $N=1,2$ superextensions; in Sect. 4 and 5 we present our partial results describing $N=1,2$ supersymmetric Poincaré and Euclidean r -matrices. More detailed plan of our paper is the following: In Sect. 2 we shall consider complex $\mathfrak{o}(4)$ and inhomogeneous $\mathfrak{io}(4)$ algebras and its $N = 1$ SUSY extension. We shall also provide their real and pseudoreal forms defining $D = 4$ Poincaré, Euclidean and Kleinian algebras and corresponding $N = 1$ real/pseudoreal superalgebras. In Sect. 3 we shall discuss complex $N = 2$ superalgebras with two central charges and odd sector described by 8 independent complex supercharges $(Q_\alpha^a, Q_{\dot{\alpha}b}, \alpha, \dot{\alpha} = 1, 2, a, b = 1, 2)$. Further following [28]–[27] and by using suitable conjugations or pseudoconjugations (real or pseudoreal forms) we shall describe $N = 2$ real Poincaré and Kleinian superalgebras as well as complex selfconjugate Euclidean superalgebras. In order to compare with earlier results for $N=0$ (nonsupersymmetric case) and $N=1$ we shall provide in Sect. 4 the tables of classical Poincaré (super)- r -matrices given in [1] and [19]². In Sect. 5 we present new results: we use the set of real Poincaré classical r -matrices provided by Zakrzewski in [1], and describe the ones which do have $N=2$ supersymmetric extension; we present also $N=2$ $D=4$ Euclidean supersymmetric r -matrices. Finally in Sect. 6 we present final remarks.

In Appendix A we shall outline the general theory of conjugations and pseudoconjugations for Lie superalgebras.

The list of $N = 2$ complex supersymmetric r -matrices and their Kleinian $\mathfrak{o}(2,2)$ real counterparts satisfying suitable (pseudo)reality conditions will be presented in our next publications.

² We point out here that among 21 classical r -matrices obtained by Zakrzewski [1] there is one class which satisfies modified YB equation. We shall consider 20 classes of r -matrices which lead to triangular deformations.

2. Complex $D=4$ Euclidean algebra, its simple ($N=1$) supersymmetrization and their real or pseudoreal forms

2.1. Complex $\mathfrak{io}(4, \mathbb{C})$ algebra and its real forms

It is known that there are four real forms of complex orthogonal Lie algebras $\mathfrak{o}(4, \mathbb{C})$ corresponding to three different nondegenerate $D = 4$ metric signatures and the fourth one which can be obtained by imposing quaternionic structure (see e.g. [29, 30]):

- i) $\mathfrak{o}(4) \cong \mathfrak{o}(3) \oplus \mathfrak{o}(3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ – Euclidean case
- ii) $\mathfrak{o}(3, 1) \cong \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ – Lorentzian case³
- iii) $\mathfrak{o}(2, 2) \cong \mathfrak{o}(2, 1) \oplus \mathfrak{o}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1) \cong \mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{sp}(2, \mathbb{R})$ – Kleinian case
- iv) $\mathfrak{o}^*(4) \cong \mathfrak{o}(3) \oplus \mathfrak{o}(2, 1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(1, 1)$ – quaternionic case.

The quaternionic origin of the fourth real form follows from isomorphism with orthogonal quaternionic algebras [29]

$$\mathfrak{o}(2; \mathbb{H}) = \mathfrak{o}^*(4), \quad \mathfrak{o}(1; \mathbb{H}) = \mathfrak{o}(2). \quad (2.1)$$

In first three cases (Euclidean, Lorentzian, Kleinian) one can lift the corresponding real forms to inhomogeneous algebra $\varepsilon(4; \mathbb{C}) = \mathfrak{o}(4; \mathbb{C}) \ltimes \mathbf{T}_{\mathbb{C}}^4$, where the complex Abelian generators describe complexified momentum fourvectors ($\mathcal{P}_{\mu} \in \mathbf{T}_{\mathbb{C}}^4$).

In fourth case one can introduce the momenta as complex $\mathfrak{o}^*(4)$ vectors described by second order fourcomponent $SU(2) \times SU(1, 1)$ complex spinors. The corresponding inhomogeneous algebra $\mathfrak{o}^*(4) \ltimes \mathbf{T}_q^4$ can be endowed with quaternionic structure if we perform the following contraction of the quaternionic symmetric coset (see also (2.1))

$$\mathfrak{o}(3; \mathbb{H}) = \mathfrak{o}(2; \mathbb{H}) \oplus \mathfrak{o}(1; \mathbb{H}) \oplus \frac{\mathfrak{o}(3; \mathbb{H})}{\mathfrak{o}(2; \mathbb{H}) \oplus \mathfrak{o}(1; \mathbb{H})} \Rightarrow (\mathfrak{o}^*(4) \oplus \mathfrak{o}(2)) \ltimes \mathbf{T}_q^4, \quad (2.2)$$

where \mathbf{T}_q^4 has eight real dimensions, i.e. the dimensionality of $\mathbf{T}_{\mathbb{C}}^4$ is not reduced. We add that the inhomogeneous $\mathfrak{o}^*(4)$ algebra does not play known significant role in the description of physically relevant $D = 4$ geometries.

The most known real form of $\mathfrak{io}(4, \mathbb{C})$ is the $D = 4$ Poincaré Lie algebra $p(3, 1) = \mathfrak{o}(3, 1) \ltimes \mathbf{T}^{1,3}$ generated by the Poincaré fourmomenta $P_{\mu} \in \mathbf{T}^{1,3}$ and six Lorentz rotations $L_{\mu\lambda} \in \mathfrak{o}(3, 1)$ ($\mu, \nu = 1, 2, 3, 4$) and looks as follows:

$$\begin{aligned} [L_{\mu\nu}, L_{\lambda\rho}] &= i(g_{\nu\lambda} L_{\mu\rho} - g_{\nu\rho} L_{\mu\lambda} + g_{\mu\rho} L_{\nu\lambda} - g_{\mu\lambda} L_{\nu\rho}), & L_{\mu\nu} &= -L_{\nu\mu}, \\ [L_{\mu\nu}, P_{\rho}] &= i(g_{\nu\rho} P_{\mu} - g_{\mu\rho} P_{\nu}), & [P_{\mu}, P_{\nu}] &= 0, \end{aligned} \quad (2.3)$$

where $g_{\mu\nu} = \text{diag}(-1, -1, -1, 1)$ is the Minkowski (Lorentzian) metric. If we replace in (2.3) such a metric by the Euclidean one i.e. $g_{\mu\nu}^E = -\delta_{\mu\nu} = \text{diag}(-1, -1, -1, -1)$, one gets the $D = 4$ Euclidean algebra $\varepsilon(4) = \mathfrak{o}(4) \ltimes \mathbf{T}^4$, described by Euclidean generators \mathcal{P}_{μ} , $\mathcal{L}_{\mu\nu}$:

$$\begin{aligned} [\mathcal{L}_{\mu\nu}, \mathcal{L}_{\lambda\rho}] &= -i(\delta_{\nu\lambda} \mathcal{L}_{\mu\rho} - \delta_{\nu\rho} \mathcal{L}_{\mu\lambda} + \delta_{\mu\rho} \mathcal{L}_{\nu\lambda} - \delta_{\mu\lambda} \mathcal{L}_{\nu\rho}), & \mathcal{L}_{\mu\nu} &= -\mathcal{L}_{\nu\mu}, \\ [\mathcal{L}_{\mu\nu}, \mathcal{P}_{\rho}] &= -i(\delta_{\nu\rho} \mathcal{P}_{\mu} - \delta_{\mu\rho} \mathcal{P}_{\nu}), & [\mathcal{P}_{\mu}, \mathcal{P}_{\nu}] &= 0. \end{aligned} \quad (2.4)$$

The Poincaré algebra can be obtained from Euclidean one by the following substitution ($r, s, t = 1, 2, 3$)

$$L_{\mu\nu} = (\mathcal{L}_{rs}, i\mathcal{L}_{0r}) \quad , \quad P_{\mu} = (\mathcal{P}_r, i\mathcal{P}_0) \quad (2.5)$$

³ The real generators $\mathfrak{o}(3, 1)$ can be described by complex-conjugated generators of $\mathfrak{o}(3; \mathbb{C})$ and $\overline{\mathfrak{o}(3; \mathbb{C})}$ (see (2.11)).

or equivalently $M_r = \mathcal{M}_r$, $N_r = -i\mathcal{N}_r$, where $M_r = \frac{1}{2}\varepsilon_{rst}L_{st}$, $\mathcal{M}_r = \frac{1}{2}\varepsilon_{rst}\mathcal{L}_{st}$ and $N_r = L_{4r}$, $\mathcal{N}_r = \mathcal{L}_{4r}$. The imaginary unit occurring in (2.5) effectively change in (2.3) the Euclidean metric $g_{\mu\nu}^E$ into the Lorentzian one $g_{\mu\nu}$.

If we choose the metric with Kleinian (neutral) signature $g_{\mu\nu}^K = \text{diag}(1, -1, 1, -1)$ one gets Lie algebra $\mathfrak{io}(2, 2) = \mathfrak{o}(2, 2) \ltimes \mathbf{T}^{2,2}$, which can be obtained by the following change of the Euclidean generators

$$M_i^K = i\mathcal{M}_i, \quad N_i^K = i\mathcal{N}_i \quad (i=1,3), \quad M_2^K = \mathcal{M}_2, \quad N_2^K = \mathcal{N}_2, \quad P_\mu^K = (\mathcal{P}_1, i\mathcal{P}_2, \mathcal{P}_3, i\mathcal{P}_4). \quad (2.6)$$

For describing the $\mathfrak{o}^*(4)$ algebra with quaternionic structure one should split the real $\mathfrak{o}(4)$ generators as follows

$$\mathfrak{o}(4) = \mathfrak{u}(2) \oplus \frac{\mathfrak{o}(4)}{\mathfrak{u}(2)} \quad (2.7)$$

and multiply the coset generators by "i".

In order to embed three cases of inhomogeneous algebras into unified framework one can consider the complex $D = 4$ Euclidean algebra $\mathcal{E}(4; \mathbb{C})$ and introduce its three real forms (Euclidean, Poincaré and Kleinian). These real forms are introduced with the help of three non-isomorphic antilinear involutive conjugations $I_A \rightarrow I_A^\#$ ($I_A \in (\mathcal{L}_{\mu\nu}, \mathcal{P}_\mu)$), where $\# = \dagger, \ddagger, \oplus$. The following reality conditions imposed on the generators of $\mathcal{E}(4; \mathbb{C})$ ($r, s = 1, 2, 3; i = 1, 3; k = 2, 4$)

$$\mathcal{L}_{rs}^\dagger = \mathcal{L}_{rs}, \quad \mathcal{L}_{4r}^\dagger = -\mathcal{L}_{4r}, \quad \mathcal{P}_r^\dagger = \mathcal{P}_r, \quad \mathcal{P}_4^\dagger = -\mathcal{P}_4 \quad \text{Poincaré case} \quad (2.8)$$

$$\mathcal{L}_{\mu\nu}^\ddagger = \mathcal{L}_{\mu\nu}, \quad \mathcal{P}_\mu^\ddagger = \mathcal{P}_\mu \quad \text{Euclidean case} \quad (2.9)$$

$$\mathcal{L}_{13}^\oplus = \mathcal{L}_{13}, \quad \mathcal{L}_{24}^\oplus = \mathcal{L}_{24}, \quad \mathcal{L}_{ik}^\oplus = -\mathcal{L}_{ik}, \quad \mathcal{P}_i^\oplus = \mathcal{P}_i, \quad \mathcal{P}_k^\oplus = -\mathcal{P}_k \quad \text{Kleinian case} \quad (2.10)$$

define respectively real $D = 4$ Poincaré, Euclidean and Kleinian algebras. We observe that first two real structures (conjugations) coincide on $\mathcal{E}(3)$ subalgebra.

Another convenient basis in complex $\mathcal{E}(4; \mathbb{C})$ algebra is obtained by introducing the pair of chiral (left-handed) and anti-chiral (right-handed) generators: $2M_r^\pm = \frac{1}{2}\varepsilon_{rst}\mathcal{L}_{st} \pm \mathcal{L}_{4r} \equiv \mathcal{M}_r \pm i\mathcal{N}_r$ describing two complex commuting $\mathfrak{o}_\pm(3, \mathbb{C}) \equiv \mathfrak{sl}_\pm(2, \mathbb{C})$ subalgebras

$$[M_r^\pm, M_s^\pm] = i\varepsilon_{ijk}M_k^\pm, \quad [M_r^\pm, M_s^\mp] = 0, \quad (2.11)$$

$$[M_r^\pm, \mathcal{P}_s] = \frac{i}{2}(\varepsilon_{rst}\mathcal{P}_t \mp \delta_{rs}\mathcal{P}_4), \quad [M_r^\pm, \mathcal{P}_4] = \pm \frac{i}{2}\mathcal{P}_r. \quad (2.12)$$

The reality conditions (2.8–2.10) imposed on the complex generators M^\pm look as follows (we provide also the fourth reality condition related with quaternionic structure; $r = 1, 2, 3; i = 1, 3$)

$$(M_r^\pm)^\dagger = M_r^\mp \quad \text{Poincaré case} \quad (2.13)$$

$$(M_r^\pm)^\ddagger = M_r^\pm \quad \text{Euclidean case} \quad (2.14)$$

$$(M_i^\pm)^\oplus = -M_i^\pm, \quad (M_2^\pm)^\oplus = M_2^\pm \quad \text{Kleinian case} \quad (2.15)$$

$$(M_r^+)^\oplus = M_r^+, \quad (M_i^-)^\oplus = -M_i^-, \quad (M_2^-)^\oplus = M_2^- \quad \text{quaternionic case} \quad (2.16)$$

Further we shall consider the supersymmetric $N=1,2$ extensions of the Euclidean and Poincaré real forms (2.13–2.15). The complexifications of real Poincaré, Euclidean and Kleinian algebras are equivalent, and one can consider as well in place of reality constraints (2.8)–(2.10) the real forms of

complexified Poincaré or Kleinian algebras in order to provide the real Poincaré, Euclidean and Kleinian algebras.

It is quite useful to work further with Lorentzian (Poincaré) canonical basis which is obtained after realification of the Cartan-Chevalley basis of $\mathfrak{sl}(2, \mathbb{C})$ ⁴. In such a basis Lorentz generators are defined as follows (see [13, 19])

$$\begin{aligned} h &= -iN_3, & e_{\pm} &= -i(N_1 \mp M_2), \\ h' &= iM_3, & e'_{\pm} &= i(M_1 \pm N_2). \end{aligned} \quad (2.17)$$

One obtains the following description of $D = 4$ Lorentz algebra

$$\begin{aligned} [h, e_{\pm}] &= \pm e_{\pm}, & [e_+, e_-] &= 2h, \\ [h, e'_{\pm}] &= \pm e'_{\pm}, & [h', e_{\pm}] &= \pm e'_{\pm}, & [e_{\pm}, e'_{\mp}] &= \pm 2h', \\ [h', e'_{\pm}] &= \mp e_{\pm}, & [e'_+, e'_-] &= -2h. \end{aligned} \quad (2.18)$$

The fourmomenta generators with the components

$$P_{\mu} = (P_1, P_2, P_{\pm} = P_4 \pm P_3) \quad (2.19)$$

extend (2.18) to the real Poincaré algebra as follows

$$\begin{aligned} [h, P_{\pm}] &= \pm P_{\pm}, & [h, P_i] &= 0 \quad (i = 1, 2), \\ [e_{\pm}, P_{\pm}] &= 0, & [e_{\pm}, P_{\mp}] &= 2P_1, & [e_{\pm}, P_1] &= P_{\pm}, & [e_{\pm}, P_2] &= 0, \end{aligned} \quad (2.20)$$

$$\begin{aligned} [h', P_{\pm}] &= 0, & [h', P_1] &= -P_2, & [h', P_2] &= P_1, \\ [e'_{\pm}, P_{\pm}] &= 0, & [e'_{\pm}, P_{\mp}] &= \mp 2P_2, & [e'_{\pm}, P_1] &= 0, & [e'_{\pm}, P_2] &= \mp P_{\pm}. \end{aligned} \quad (2.21)$$

Three reality conditions imposed on these canonical generators are now

$$x_A^{\dagger} = -x_A \quad \text{for } x_A \in (h, e_{\pm}, h', e'_{\pm}, P_{\pm}, P_1, P_2) \quad (\text{Poincaré case}) \quad (2.22)$$

$$\begin{aligned} h^{\dagger} &= h, & e_{\pm}^{\dagger} &= e_{\mp}, & h'^{\dagger} &= -h', & e'^{\dagger}_{\pm} &= -e'_{\mp}, \\ P_{\pm}^{\dagger} &= -P_{\mp}, & P_i^{\dagger} &= P_i \quad (i = 1, 2) \end{aligned} \quad (\text{Euclidean case}) \quad (2.23)$$

$$\begin{aligned} h^{\oplus} &= -h, & e_{\pm}^{\oplus} &= -e_{\pm}, & h'^{\oplus} &= h', & e'^{\oplus}_{\pm} &= e'_{\pm}, \\ P_{\pm}^{\oplus} &= -P_{\pm}, & \tilde{P}_{\pm}^{\oplus} &= -\tilde{P}_{\pm}^{\oplus}, & \tilde{P}_{\pm} &= P_1 \pm iP_2 \end{aligned} \quad (\text{Kleinian case}) \quad (2.24)$$

2.2. Complex $D = 4$ $N = 1$ Euclidean superalgebra

In this paper we shall consider the superalgebra generators in purely algebraic way, without reference to concrete realizations. In this subsection we shall recall the complex $D = 4$ $N = 1$ Euclidean superalgebra (see e.g. [31, 32, 18]) describing simple ($N = 1$) supersymmetrization of $\varepsilon(4; \mathbb{C})$ (inhomogeneous $\mathfrak{o}(4; \mathbb{C})$) complex algebra. Such superalgebra is obtained by adding to the generator of complex $D = 4$ Euclidean algebra $\varepsilon(4; \mathbb{C}) = \mathfrak{o}(4; \mathbb{C}) \ltimes \mathbf{T}_4^{\mathbb{C}}$ four independent complex supercharges $Q_{\alpha}, \bar{Q}_{\dot{\alpha}}$ transforming

⁴ Let (h, e_{\pm}) be the Cartan-Weyl basis of $\mathfrak{sl}(2, \mathbb{C})$ with the commutation relations in the first line of (2.18). Setting $h' := ih, e'_{\pm} := ie_{\pm}$ we obtain all commutation relations (2.18). The real Lie algebra $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ generated by the elements $(h, e_{\pm}, h', e'_{\pm})$ with the defining relations (2.17) is called realification of $\mathfrak{sl}(2, \mathbb{C})$.

as fundamental representations under “left” and “right” internal symmetry groups $SL_+(2;C)$ and $SL_-(2;C)$ ⁵. The supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ extend the $D=4$ complex Euclidean algebra (2.4) by the following algebraic relations

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma_\mu^E)_{\alpha\dot{\beta}} \mathcal{P}^\mu \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad (2.25)$$

$$[\mathcal{L}_{\mu\nu}, Q_\alpha] = -(\sigma_{\mu\nu}^E)_\alpha{}^\beta Q_\beta \quad [\mathcal{L}_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = \bar{Q}_{\dot{\beta}} (\tilde{\sigma}_{\mu\nu}^E)^{\dot{\beta}}{}_{\dot{\alpha}} \quad (2.26)$$

$$[\mathcal{P}_\mu, Q_\alpha] = [\mathcal{P}_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \quad (2.27)$$

where the Euclidean sigma matrices σ_μ^E are expressed by standard Pauli matrices σ_r ($r=1,2,3$) as follows

$$(\sigma_\mu^E)_{\alpha\dot{\beta}} = ((\sigma_r)_{\alpha\dot{\beta}}, i(I_2)_{\alpha\dot{\beta}}) \quad r=1,2,3 \quad (2.28)$$

and satisfy the known reality conditions under Hermitean matrix conjugation

$$(\sigma_r^E)^+ = \sigma_r^E \quad (\sigma_0^E)^+ = -\sigma_0^E. \quad (2.29)$$

The matrices $\sigma_{\mu\nu}^E$ and $\tilde{\sigma}_{\mu\nu}^E$ describe the following realizations of the pair of commuting $sl_\pm(2;C)$ algebras

$$\begin{aligned} sl_+(2;C) : \quad (\sigma_{\mu\nu}^E)_\alpha{}^\beta &= (\sigma_{rs}^E = \frac{1}{2}\epsilon_{rst}\sigma_t, \quad \sigma_{0r}^E = \frac{1}{2}\sigma_r), \\ sl_-(2;C) : \quad (\tilde{\sigma}_{\mu\nu}^E)_{\dot{\alpha}}{}^{\dot{\beta}} &= (\tilde{\sigma}_{rs}^E = \frac{1}{2}\epsilon_{rst}\sigma_t, \quad \tilde{\sigma}_{0r}^E = -\frac{1}{2}\sigma_r). \end{aligned} \quad (2.30)$$

The complexified Euclidean fourmomenta $\mathcal{P}_{\alpha\dot{\beta}} \equiv (\sigma_\mu^E)_{\alpha\dot{\beta}} \mathcal{P}^\mu$ transform under the complex Euclidean rotations $A_\nu^\mu \in O(4;C)$ described by the product of two commuting Lorentz groups $SL_\pm(2;C)$ as follows ($S_\alpha{}^\beta \in SL_+(2;C)$, $\tilde{S}_{\dot{\alpha}}{}^{\dot{\beta}} \in SL_-(2;C)$)

$$\mathcal{P}'_{\alpha\dot{\beta}} = S_\alpha{}^\gamma \tilde{S}_{\dot{\beta}}{}^{\dot{\delta}} \mathcal{P}_{\gamma\dot{\delta}} \longleftrightarrow \mathcal{P}'_\mu = A_\mu^\nu \mathcal{P}_\nu \quad (2.31)$$

where $\mathcal{P}_\mu = \frac{1}{2}\sigma_\mu^{\alpha\dot{\beta}} \mathcal{P}_{\alpha\dot{\beta}}$ and $A_\nu^\mu = \sigma_\nu^{\alpha\dot{\beta}} \sigma_\mu^{\gamma\dot{\delta}} S_\alpha{}^\gamma \tilde{S}_{\dot{\beta}}{}^{\dot{\delta}}$.

Finally one can check that the relations (2.25–2.27) are invariant under the complex rescaling transformation (c is a complex number)

$$Q'_\alpha = c Q_\alpha \quad \bar{Q}'_{\dot{\alpha}} = c^{-1} \bar{Q}_{\dot{\alpha}}. \quad (2.32)$$

The rescaling (2.32) is described by $GL(1;C) = U(1) \times R$ Abelian group and represents the one-parameter complex internal symmetries of $D=4$ simple complex Euclidean superalgebra. The $N=1$ internal $gl(1;C)$ generator T satisfies the algebraic relation

$$[T, Q_\alpha] = Q_\alpha \quad [T, \bar{Q}_{\dot{\alpha}}] = -\bar{Q}_{\dot{\alpha}}. \quad (2.33)$$

⁵ We recall that $O(4;C) = O_+(3;C) \otimes O_-(3;C)$ has a spinorial covering $SL_+(2;C) \otimes SL_-(2;C)$. The $SL_+(2;C)$ spinors have undotted spinorial indices, and dotted indices characterize $SL_-(2;C)$ spinors. These two groups are also called chiral (left) and antichiral (right) projections of $o(4;C)$ group.

2.3. Real and pseudoreal forms of $D = 4$ $N = 1$ complex Euclidean superalgebra.

2.3.1. $D = 4$ $N = 1$ pseudoreal Euclidean superalgebra and its pseudoreal forms. The reality conditions (2.9) defining $D = 4$ Euclidean space-time algebra $\mathfrak{o}(4; \mathbb{R}) \ltimes \mathbf{T}_4$ can be extended to the sector of Euclidean spinorial supercharges if we use the corresponding spinorial covering group $\overline{O}(4; \mathbb{R}) = SU_+(2) \oplus SU_-(2)$ which requires a pair of independent two-component complex $SU(2)$ spinors. This property of doubling of $D = 4$ Euclidean spinorial components in comparison with standard relativistic $D = 4$ case leads to the known conclusion that contrary to the Poincaré case the four-component real (Majorana) Euclidean spinors do not exist. In the algebraic framework one can however extend in odd supercharges sector the conjugation (2.9) as the pseudoconjugation (see Appendix A, (A.3b)) which should be consistent with $N = 1$ complex Euclidean algebra (2.25–2.27) under the assumption that the generators $\mathcal{P}_\mu, \mathcal{L}_{\mu\nu}$ are real Euclidean, i.e. satisfy the reality conditions (2.9). The $N = 1$ pseudoconjugation of Euclidean supercharges is an involution of fourth order (see also (A.3b)) satisfying the relation

$$(Q_\alpha^\dagger)^\dagger = -Q_\alpha, \quad (\bar{Q}_\alpha^\dagger)^\dagger = -\bar{Q}_\alpha \quad (2.34)$$

and it look as follows [24, 22, 25]

$$Q_\alpha^\dagger = \varepsilon_{\alpha\beta} Q_\beta, \quad \bar{Q}_\alpha^\dagger = \eta \varepsilon_{\alpha\beta} \bar{Q}_\beta, \quad \eta = \pm 1. \quad (2.35)$$

It can be shown that the map $Q_\alpha \rightarrow -\varepsilon_{\alpha\beta} Q_\beta^\dagger, \bar{Q}_\alpha \rightarrow -\eta \varepsilon_{\alpha\beta} \bar{Q}_\beta^\dagger$ leaves the superalgebra (2.25–2.27) invariant provided that we choose the parameter q (see Appendix, A (A.1)) in accordance with the relation

$$\eta = (-1)^{q+1}. \quad (2.36)$$

We obtain the following two cases:

i) $\eta = -1, q = 0$ (standard choice)

In this case the product of odd (fermionic operators f_1, f_2 which are odd powers of supercharges are conjugated as follows

$$(f_1 f_2)^\dagger = f_2^\dagger f_1^\dagger \quad (2.37)$$

and leads to standard supersymmetry scheme with pseudoconjugation \ddagger which can be represented on complex Grassmannian variables $\theta_\alpha = \theta_\alpha^1 + i\theta_\alpha^2$ as the complex conjugation $\theta_\alpha \rightarrow \theta_\alpha^* = \theta_\alpha^1 - i\theta_\alpha^2$. It should be added that all applications of pseudoconjugations to the description of real $N = 1$ Euclidean SUSY use the case i) (see e.g. [23]).

ii) $\eta = 1, q = 1$ (exotic choice)

In this case the product of odd (fermionic) operators f_1, f_2 which are odd powers of supercharges and Grassmann variables are conjugated as follows

$$(f_1 f_2)^\dagger = -f_2^\dagger f_1^\dagger. \quad (2.38)$$

Such choice leads to nonstandard supersymmetry scheme, which can be realized in complex superspace described by odd Grassmann variables $\tilde{\theta}_A, \bar{\tilde{\theta}}_A$ with the products transforming under conjugation in the nonstandard way

$$(\tilde{\theta}_A \bar{\tilde{\theta}}_B)^\dagger = -\bar{\tilde{\theta}}_B^\dagger \tilde{\theta}_A^\dagger, \quad (\bar{\tilde{\theta}}_A \tilde{\theta}_B)^\dagger = -(\bar{\tilde{\theta}})^\dagger_B (\tilde{\theta})^\dagger_A. \quad (2.39)$$

Such type of Grassmann variables was considered as mathematically consistent choice in [26, 27, 28] but it has not been applied in the literature to describe the physical supersymmetric systems, therefore exotic.

In the exotic case the conjugation $A \mapsto A^\ddagger$ (antilinear antiautomorphism of second order) in bosonic sector of the Euclidean superalgebra defining the reality condition (2.9) is lifted in fermionic sector of supercharges to the antilinear antiautomorphism of fourth order defining pseudoconjugation.

One can show that for both values $\eta = \pm 1$ in the scaling transformations (2.32) the parameter c should be real, what means that the invariance of (2.33) under the pseudoconjugation (2.35) is valid if $T^\dagger = -T$, i.e. $N = 1$ internal symmetry $GL(1; \mathbb{C}) = U(1) \oplus O(1, 1)$ is reduced to $GL(1; \mathbb{R}) = O(1, 1)$ for pseudoreal $N = 1$ Euclidean superalgebra.

2.3.2. $D = 4$ $N = 1$ real Poincaré superalgebra $N = 1$ Poincaré superalgebra is obtained from the relations (2.25–2.27) after imposing the following extension of the Poincaré reality condition (2.8) to the supercharges sector

$$(Q_\alpha)^\dagger = \bar{Q}_{\dot{\alpha}}, \quad (\bar{Q}_{\dot{\beta}})^\dagger = \eta Q_\beta \quad \eta = \pm 1. \quad (2.40)$$

If $\eta = 1$, $q = 0$ the formula (2.40) describe conjugation and lead to standard $N = 1$ real Poincaré supersymmetry. If $\eta = -1$ we get $q = 1$ and relations (2.40) describe pseudoconjugation (pseudoreality condition) defining exotic supersymmetry with Grassmann variables satisfying relations (2.38).

After introducing $P_i = \mathcal{P}_i$, $P_0 = i\mathcal{P}_0$ (see 2.5) one gets from (2.25–2.27) the well-known $N=1$ Poincaré superalgebra with the pair of supercharges represented by two-component complex Weyl spinors

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma_\mu)_{\alpha\dot{\beta}} P^\mu, & [P_\mu, Q_\alpha] &= [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \\ \{Q_\alpha, Q_\beta\} &= 0, & \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0, \\ [L_{\mu\nu}, Q_\alpha] &= -(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, & [L_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] &= \bar{Q}_{\dot{\beta}a}(\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}, \end{aligned} \quad (2.41)$$

where the Lorentzian σ -matrices are defined as follows

$$\begin{aligned} \sigma_\mu &= (\sigma_i = \sigma_i^E, \quad \sigma_4 = i\sigma_4^E), \\ \sigma_{\mu\nu} &= (\sigma_{ij} = \sigma_{ij}^E, \quad \sigma_{4i} = -i\sigma_{4i}^E), \\ \tilde{\sigma}_{\mu\nu} &= (\tilde{\sigma}_{ij} = \tilde{\sigma}_{ij}^E, \quad \tilde{\sigma}_{4i} = -i\tilde{\sigma}_{4i}^E). \end{aligned} \quad (2.42)$$

It is easy to see that for $\eta = \pm 1$ the pair of relations (2.33) is invariant under the conjugation (2.40) if $T^\dagger = T$, i.e. the internal symmetry of $N = 1$ Poincaré supersymmetry is described by the restriction of $GL(1; \mathbb{C})$ to $U(1)$.

2.3.3. $D = 4$ $N = 1$ real Kleinian superalgebra The $N = 1$ Kleinian $\mathfrak{o}(2, 2)$ superalgebra is obtained after the following extension of the reality condition (2.10) to the supercharges sector

$$(Q_\alpha)^\oplus = Q_\alpha, \quad (\bar{Q}_{\dot{\beta}})^\oplus = \eta \bar{Q}_{\dot{\beta}} \quad \eta = \pm 1, \quad (2.43)$$

i.e. the supercharges (2.43) form a pair of respectively $\mathfrak{o}_+(2, 1) = \mathfrak{sl}_+(2; \mathbb{R})$ and $\mathfrak{o}_-(2, 1) = \mathfrak{sl}_-(2; \mathbb{R})$ real spinors. Imposing on complex superalgebra (2.25–2.27) the reality condition (2.43) one gets the following real superalgebra

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\tilde{\sigma}_\mu)_{\alpha\dot{\beta}} \tilde{P}^\mu, \quad (2.44)$$

where $\tilde{P}_\mu \tilde{P}^\mu = \tilde{P}_1^2 - \tilde{P}_2^2 + \tilde{P}_3^2 - \tilde{P}_4^2$ and

$$\tilde{P}_\mu = (\mathcal{P}_1, i\mathcal{P}_2, \mathcal{P}_3, i\mathcal{P}_4) \quad (2.45)$$

with $\tilde{\sigma}_\mu$ describing $\mathfrak{o}(2, 2)$ real σ -matrices

$$\tilde{\sigma}_\mu = (\sigma_1, i\sigma_2, \sigma_3, -I_2). \quad (2.46)$$

The inhomogeneous $\mathfrak{o}(2, 2)$ algebra can be described more conveniently if we use the formulation of complex Euclidean algebra $\varepsilon(4; \mathbb{C})$ using formulae (2.11–2.12). In Kleinian case the "complex"

generators M_r^\pm become the set of real generators (i.e. generating real Lie superalgebra with real structure constants)

$$\begin{aligned} [M_r^{(+)}, Q_\alpha^a] &= -\frac{1}{2}(\tilde{\sigma}_r)_\alpha{}^\beta Q_\beta^a, & [M_r^{(-)}, Q_\alpha^a] &= 0, \\ [M_r^{(+)}, \bar{Q}_{\dot{\alpha}a}] &= 0, & [M_r^{(-)}, \bar{Q}_{\dot{\alpha}a}] &= \frac{1}{2}\bar{Q}_{\dot{\beta}a}(\tilde{\sigma}_r)^{\dot{\beta}}{}_{\dot{\alpha}}. \end{aligned} \quad (2.47)$$

We add that similarly as in Poincaré case, the choice $\eta = -1$, $q = 0$ leads to standard $N = 1$ Kleinian supersymmetry with (2.43) describing conjugation \oplus (reality conditions), and if $\eta = 1$, $q = 1$ one gets (see also Sec. 2.3.1) exotic supersymmetry with pseudoconjugation implying the exotic antiautomorphism which leads to relations (2.38). Finally for both values of η the relations (2.33) are consistent with the reality condition (2.43) if $T^\oplus = -T$, i.e. real $N = 1$ Kleinian symmetry is endowed with $GL(1; \mathbb{R}) \equiv O(1, 1)$ internal symmetry.

3. Complex $D = 4$ $N = 2$ Euclidean supersalgebra and its Euclidean, Poincaré and Kleinian (pseudo) real forms

3.1. Complex $D = 4$ $N = 2$ Euclidean superalgebra

In this subsection we shall describe $N = 2$ superization of complex inhomogeneous $\mathfrak{o}(4; \mathbb{C})$ algebra. Basic $N=1$ relations (2.25) are extended to $N = 2$ as follows ($a, b.. = 1, 2$):

$$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = 2\delta_b^a (\sigma_\mu^E)_{\alpha\dot{\beta}} \mathcal{P}^\mu, \quad (3.1)$$

$$\{Q_\alpha^a, Q_\beta^b\} = \varepsilon_{\alpha\beta} \varepsilon^{ab} Z, \quad \{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} = \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{ab} \tilde{Z}, \quad (3.2)$$

where Z, \tilde{Z} describe a pair of complex scalar central charges. The relations (3.1) are invariant under the internal symmetries $A \in GL(2, \mathbb{C})$, where

$$Q_\alpha^a \rightarrow (A)^a{}_b Q_\alpha^b, \quad \bar{Q}_{\dot{\alpha}a} \rightarrow \bar{Q}_{\dot{\alpha}b} (A^{-1})^b{}_a. \quad (3.3)$$

The presence of central charges breaks only internal symmetry $GL(2; \mathbb{C})$ to $SL(2; \mathbb{C})$, what follows from the relation $A \varepsilon A^T = \varepsilon$ valid for complex 2×2 matrices $A \in SL(2; \mathbb{C})$ due to the equivalence $SL(2; \mathbb{C}) \sim Sp(2; \mathbb{C})$. The $\mathfrak{gl}(2; \mathbb{C})$ generators T_i^j ($i, j = 1, 2$)

$$[T_i^j, T_k^l] = i(\delta_k^j T_i^l - \delta_i^l T_k^j) \quad (3.4)$$

are restricted to $\mathfrak{sl}(2; \mathbb{C})$ by the condition $T_0 \equiv T_1^1 + T_2^2 = 0$.

One can describe $N = 2$ internal symmetries algebra $\mathfrak{gl}(2; \mathbb{C})$ in convenient way by replacing four generators T_i^j by generators T_A ($A = 0, 1, 2, 3$) adjusted to further description of $U(2)$ internal symmetry

$$T_A = \frac{i}{2} (\sigma_A)^i{}_j T_i^j, \quad (3.5)$$

where $\sigma_A = (\sigma_r, \sigma_0 = I_2)$; σ_r are three 2×2 Hermitean Pauli matrices. The $\mathfrak{gl}(2; \mathbb{C})$ covariance relation of $N = 2$ supercharges looks as follows

$$[T_A, Q_\alpha^a] = -\frac{1}{2} (\sigma_A)^a{}_b Q_\alpha^b, \quad [T_A, \bar{Q}_{\dot{\alpha}a}] = \frac{1}{2} \bar{Q}_{\dot{\alpha}b} (\sigma_A)^b{}_a, \quad (3.6)$$

where the 2×2 matrices $(t_A)^b{}_a = \frac{1}{2} (\sigma_A)^b{}_a$ provide the fundamental realizations of $\mathfrak{gl}(2, \mathbb{C})$ generators (3.5)⁶. Using known relation ($r, s, t = 1, 2, 3$)

$$\sigma_r \sigma_s = \delta_{rs} + i \varepsilon_{rst} \sigma_t \quad (3.7)$$

⁶ It corresponds to the realization $(T_i^j)^b{}_a = i \delta_i^b \delta_a^j$ of the generators in the relations (3.4).

one gets from (3.4) and (3.5) the $N = 2$ internal symmetry algebra $\mathfrak{gl}(2)$

$$[T_r, T_s] = i\epsilon_{rst}T_t \quad [T_0, T_r] = 0. \quad (3.8)$$

Three generators T_r span $\mathfrak{sl}(2; \mathbb{C})$ algebra which is preserved even in the case of nonvanishing central charges Z, \bar{Z} ; the Abelian generator T_0 describing coset $\frac{GL(2; \mathbb{C})}{SL(2; \mathbb{C})}$ is broken in the presence of central charges.

If we consider the chiral projections of Euclidean $\mathfrak{o}(4; \mathbb{C})$ generators (see (2.11)–(2.12)) one gets from (3.6) the commutators exposing the chiral (antichiral) nature of supercharges Q_α^i ($\bar{Q}_{\dot{\alpha}i}$)

$$\begin{aligned} [M_r^{(+)}, Q_\alpha^a] &= -\frac{1}{2}(\sigma_r)_\alpha{}^\beta Q_\beta^a, & [M_r^{(-)}, Q_\alpha^a] &= 0, \\ [M_r^{(+)}, \bar{Q}_{\dot{\alpha}a}] &= 0, & [M_r^{(-)}, \bar{Q}_{\dot{\alpha}a}] &= \frac{1}{2}\bar{Q}_{\dot{\beta}a}(\sigma_r)^\beta{}_{\dot{\alpha}}. \end{aligned} \quad (3.9)$$

The vanishing commutators in (3.9) illustrate that the supercharges Q_α^a are left-handed (chiral) and the supercharges $\bar{Q}_{\dot{\alpha}a}$ are right-handed (antichiral).

3.2. $N = 2$ real Poincaré superalgebras with central charges

The reality conditions for supercharges can take the form (see e.g. [33, 34]):

$$(Q_\alpha^a)^\dagger = \bar{Q}_{\dot{\alpha}a}, \quad (\bar{Q}_{\dot{\beta}b})^\dagger = \eta Q_\beta^b \quad \eta = \pm 1. \quad (3.10)$$

The reality condition for well-known $N = 2$ real Poincaré superalgebra with central charges [33] is obtained if we put $q = 0$ (see (A.1)) and $\eta = 1$. In such a case we impose on the complex generators $\{\mathcal{L}_{\mu\nu}, \mathcal{P}_\mu, Q_\alpha^i, \bar{Q}_{\dot{\beta}j}, T_r = \frac{1}{2}(\sigma_r)^i{}_j T_i^j, Z_1, Z_2; i = 1, 2; r = 1, 2, 3\}$ of centrally extended $N = 2$ complex Euclidean superalgebra the reality constraints which extend consistently the conjugation (2.4) in bosonic sector to odd superalgebra generators.

In particular in the representation which permits the Hermitean conjugation of supercharges the conjugation (3.10) can be seen as Hermitean conjugation. Further the reality constraints on the internal symmetry generators T_A (see (3.5) and (3.8)) and the central charges (Z_1, Z_2) , which are consistent with the relations (3.1–3.2), (3.6) and (3.8) are the following

$$T_r^\dagger = T_r, \quad T_0^\dagger = -T_0, \quad Z_1^\dagger = Z_2, \quad Z_2^\dagger = Z_1, \quad (3.11)$$

where the generator T_0 describes internal symmetry only in the case when central charges vanish. If the central charges are not vanishing from first set of the relations (3.11) one can see that the algebra $\mathfrak{sl}(2; \mathbb{C})$ is constrained to $\mathfrak{su}(2)$ algebra. If we use the formulae (2.42) for Minkowskian σ -matrices, it follows from (2.5) and (2.28) that $\sigma_\mu P^\mu = \sigma_\mu^E P^{E\mu}$ and we get the following real $N = 2$ Poincaré superalgebra with one complex central charge Z :

$$\begin{aligned} \{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} &= 2\delta_b^a(\sigma_\mu)_{\alpha\dot{\beta}}P^\mu, \\ \{Q_\alpha^a, Q_\beta^b\} &= \epsilon^{ab}\epsilon_{\alpha\beta}Z, & \{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} &= \epsilon_{ab}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{Z}, \\ [L_{\mu\nu}, Q_\alpha^a] &= -(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^a, & [L_{\mu\nu}, \bar{Q}_{\dot{\alpha}a}] &= \bar{Q}_{\dot{\beta}a}(\tilde{\sigma}_{\mu\nu})^\beta{}_{\dot{\alpha}}, \\ [T_r, Q_\alpha^a] &= -(t_r)^a{}_b Q_\alpha^b, & [T_r, \bar{Q}_{\dot{\alpha}a}] &= \bar{Q}_{\dot{\alpha}b}(t_r)^b{}_a, \\ [P_\mu, Q_\alpha^a] &= [P_\mu, \bar{Q}_{\dot{\alpha}a}] = 0, & [T_r, T_s] &= i\epsilon_{rst}T_m. \end{aligned} \quad (3.12)$$

If $Z \neq 0$ the $N = 2$ internal symmetries (R -symmetries) are described by $\mathfrak{su}(2)$ algebra, with the fundamental 2×2 matrix realizations (see (2.25) and (3.12) (the fourth line)) described by Hermitean

Pauli matrices ($t_r \equiv \sigma_r$). If $Z = 0$ in relations (3.12) one can add fourth R -symmetry generator T_0 describing the extension of $\mathfrak{su}(2)$ to $\mathfrak{u}(2)$. In such a case the complex $GL(1; \mathbb{C})$ rescaling in complex $N=2$ Euclidean superalgebra generated by T_0 (see (3.5) and (3.8)); is restricted to $U(1)$ phase transformations

$$Q_\alpha^l{}' = (\exp^{i\xi}) Q_\alpha^l, \quad \bar{Q}_{\beta i}' = (\exp^{-i\xi}) \bar{Q}_{\beta i}. \quad (3.13)$$

If $\eta = -1$ the map (3.10) describes a pseudoconjugation, but requires the exotic version ($q = 1$) of formula (A.1), i.e. we get the following relation between η in (3.10) and parameters q (see (A.1))

$$\eta = (-1)^q. \quad (3.14)$$

Further it can be shown that the antiautomorphism (3.10) leads to the following reality constraints on central supercharges Z, \tilde{Z}

$$Z^\dagger = -\eta \tilde{Z}, \quad \tilde{Z}^\dagger = -\eta Z. \quad (3.15)$$

The antiautomorphism of complex $N = 2$ relations (see (3.1)–(3.8)) leads for both cases $\eta = \pm 1$ to the same restriction of internal $gl(2, \mathbb{C})$ to its subgroup: $SU(2)$ in the presence of central charge ($Z \neq 0$) and $U(2)$ if $Z = 0$. We see therefore that we obtain the same internal symmetry sectors for the standard conjugation ($\eta = 1$) and nonstandard pseudoconjugations ($\eta = -1$).

3.3. $N = 2$ pseudoconjugations and corresponding selfconjugate $N = 2$ Euclidean superalgebras

We shall consider now the $N = 2$ complex Euclidean superalgebra with the supercharges Q_α^a and $\bar{Q}_{\dot{\alpha} a}$ (see Sect. 3.1). The pseudoconjugation map (2.35) for $N = 1$ superalgebra can be extended to $N = 2$ as follows (see also [23, 24, 25]):

$$(Q_\alpha^a)^\dagger = \varepsilon_{\alpha\beta} Q_\beta^a, \quad (\bar{Q}_{\dot{\alpha} a})^\dagger = \eta \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\beta} a}, \quad \eta = \pm 1. \quad (3.16)$$

In order to obtain the antiautomorphism of $N = 2$ complex superalgebra (see Sec. 3.1) we should postulate the relation (2.36) implying that $q = 0$ (see (A.1)) for $\eta = -1$ and $q = 1$ for $\eta = 1$. We get therefore two prolongations to fermionic sector of the Euclidean conjugation (2.8). The invariance of the relation (3.1–3.2) under the pseudoconjugation maps (3.16) implies the following reality conditions for central charges Z, \tilde{Z}

$$Z^\dagger = (-1)^{q+1} Z, \quad \tilde{Z}^\dagger = (-1)^{q+1} \tilde{Z} \quad (3.17)$$

i.e. we obtain two real ($q = 1$) or imaginary ($q = 0$) central charges.

It is easy to show that the $N = 2$ supersymmetrization of complex $\mathfrak{io}(4; \mathbb{C})$ algebra becomes selfconjugate under pseudoconjugation (3.16) under the assumption that the Euclidean fourmomentum generators \mathcal{P}_μ are real (see (2.9)). Further, using the relations $\varepsilon \bar{\sigma}_i \varepsilon = \sigma_i$, we should choose for $\mathfrak{o}(4, \mathbb{C})$ generators the Euclidean reality conditions. The invariance of superalgebra under the pseudoconjugation (3.16) requires that (see (3.12))

$$T_i^j = (T_i^j)^\dagger \quad (t_i^j)^d_c = -(\bar{t}_i^j)^d_c, \quad (3.18)$$

where $t \mapsto \bar{t}$ is the complex conjugation of 2×2 matrix elements describing fundamental realizations of $gl(2; \mathbb{C})$ generators T_i^j . If central charges are absent the reality constraints (3.18) restrict the complex internal $gl(2; \mathbb{C})$ algebra to its real form $gl(2; \mathbb{R})$. The presence of scalar central charges Z, \tilde{Z} commuting with Euclidean $\mathfrak{o}(4)$ and internal symmetry generators reduces the internal symmetries $GL(2; \mathbb{R})$ to its subgroup $SL(2; \mathbb{R})$; i.e. the $\mathfrak{sl}(2; \mathbb{R})$ subalgebra of $gl(2; \mathbb{R})$ remains not broken for any value of central charges (3.17) and describes $N=2$ Euclidean R -symmetry.

3.4. $N = 2$ conjugation and corresponding $N = 2$ real Euclidean superalgebra

If $N = 2$ it is possible to introduce as well the following conjugation

$$(Q_\alpha^a)^\dagger = \varepsilon_{\alpha\beta} \varepsilon^{ab} Q_\beta^b, \quad (\bar{Q}_{\dot{\alpha}a})^\dagger = \eta \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{ab} \bar{Q}_{\dot{\beta}b}, \quad \eta = \pm 1 \quad (3.19)$$

with involutive property

$$((Q_\alpha^a)^\dagger)^\dagger = Q_\alpha^a, \quad ((\bar{Q}_{\dot{\alpha}a})^\dagger)^\dagger = \bar{Q}_{\dot{\alpha}a}, \quad (3.20)$$

which is an antilinear antiautomorphism describing the superextension of conjugation (2.9). It can be checked that (3.19) describes conjugation if $\eta = 1$ and pseudoconjugation if $\eta = -1$. Due to the relation if $\eta = (-1)^q$ if $\eta = 1$ (3.19) provides standard antiautomorphism of complex $N = 2$ superalgebra with $q = 0$; if $\eta = -1$ one should postulate nonstandard antiautomorphism (see (A.1)) with $q = 1$. The restrictions on $N = 2$ central charges Z, \tilde{Z} which follow from the isomorphism of $N = 2$ Euclidean superalgebra under the (pseudo)conjugation (3.19) are the following

$$Z^\dagger = \eta Z \quad \tilde{Z}^\dagger = \eta \tilde{Z}, \quad \eta = \pm 1. \quad (3.21)$$

The covariance of relations (3.6) under the conjugation (3.19) implies that the generators T_A of internal symmetry $\mathfrak{gl}(2, C)$ satisfy the reality conditions ($A = r, 0$)

$$(T_r)^\dagger = T_r \quad T_0^\dagger = -T_0. \quad (3.22)$$

One can express in $N = 2$ case the relations (3.6) consistently with (3.19) as follows

$$[T_0, Q_\alpha^a] = -\frac{1}{2} Q_\alpha^a \quad [T_r, Q_\alpha^a] = -\frac{1}{2} (\sigma_r)^a_b Q_\alpha^b. \quad (3.23)$$

The generators T_r and T_0 due to relations (3.22) and (3.5) describe the internal $N = 2$ algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1) = \mathfrak{u}(2)$ (see also e.g. [25]).

In the realizations of superalgebra permitting to define the Hermitean conjugation of supercharges $Q_\alpha^a \rightarrow Q_{\dot{\alpha}a}^*$, $\bar{Q}_{\dot{\alpha}a} \rightarrow \bar{Q}_{\dot{\alpha}a}^{a*}$ the antilinearity of the automorphism (3.19) can be realized explicitly. If $\eta = 1$ (i.e. $q = 0$) one can introduce the following counterpart of the conjugation (3.19) which employs the Hermitean conjugation

$$(Q_\alpha^a)^\dagger = \varepsilon_{\alpha\beta} \varepsilon^{ab} Q_\beta^{b*} \quad (\bar{Q}_{\dot{\alpha}a})^\dagger = \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{ab} \bar{Q}_{\dot{\beta}b}^*. \quad (3.24)$$

In such a case one can define the following two complex-conjugated $N=2$ Euclidean superalgebras [32]:

- holomorphic $N=2$ Euclidean superalgebra $\mathcal{E}(4; 2|\mathbb{C})$ generated by supercharges $Q_\alpha^a, \bar{Q}_{\dot{\alpha}a}$
- antiholomorphic $N=2$ Euclidean superalgebra $\overline{\mathcal{E}(4; 2|\mathbb{C})}$ generated by Hermitean – conjugated supercharges $(Q_\alpha^a)^*, (\bar{Q}_{\dot{\alpha}a})^*$.

The reality conditions described by conjugation (3.24) maps $\mathcal{E}(4; 2|\mathbb{C}) \longrightarrow \overline{\mathcal{E}(4; 2|\mathbb{C})}$, i.e. they describe the inner automorphism of real $N=2$ Euclidean superalgebra $\mathcal{E}(4; 2|\mathbb{C}) \oplus \overline{\mathcal{E}(4; 2|\mathbb{C})}$. Such form of $N=2$ Euclidean reality conditions has been employed in earlier applications, e.g. for the description of $N=2$ Euclidean supersymmetric field-theoretic models, formulated in complex $N=2$ superspace (see e.g. [36, 32, 37])

3.5. $N = 2$ real Kleinian superalgebra with central charges

For $N = 2$ Kleinian real supersymmetry we have the following possible reality conditions:

$$(Q_\alpha^a)^\oplus = Q_\alpha^a, \quad (\bar{Q}_{\dot{\alpha}a})^\oplus = \eta \bar{Q}_{\dot{\alpha}a}, \quad \eta = \pm 1 \quad (3.25)$$

which extend from $N = 1$ to $N=2$ Kleinian reality condition (2.43). The $N = 2$ Kleinian superalgebra for $q = 0$ because of the condition $\eta = (-1)^q$ takes the following form (see also (2.45–2.46))

$$\{Q_\alpha^a, \bar{Q}_{\beta b}\} = 2\delta_b^a (\tilde{\sigma}_\mu)_{\alpha\beta} \tilde{\mathcal{P}}^\mu \quad (3.26)$$

$$\{Q_\alpha^a, Q_\beta^b\} = \epsilon_{\alpha\beta} \epsilon^{ab} Z, \quad \{\bar{Q}_{\alpha a}, \bar{Q}_{\beta b}\} = \epsilon_{\alpha\beta} \epsilon_{ab} \tilde{Z}, \quad (3.27)$$

where the reality conditions for two central charges look as follows

$$Z^\oplus = \eta Z \quad \tilde{Z}^\oplus = \eta \tilde{Z}. \quad (3.28)$$

The internal symmetry generator T_A ($A = 0, r$) (see (3.5), (3.8)) satisfy for vanishing central charges the reality condition

$$(T_A)^\oplus = T_A.$$

If central charges do not vanish the R-symmetry $GL(2; \mathbb{R})$ is reduced to $SL(2; \mathbb{R})$.

4. Classical real Poincaré and Euclidean r-matrices and their N=1 superextensions

4.1. General remarks

We shall follow the method used in our previous paper [19] based on the following steps:

i) Consider Zakrzewski list of 21 real classical r -matrices satisfying classical Yang-Baxter (YB) ⁷, and use in their presentation the canonical Poincaré basis (Sec. 2.1);

ii) Remove the Poincaré reality conditions (see (2.4) or (2.17)) imposed in [1]. The generators $h, e_\pm, h', e'_\pm, P_1, P_2, P_\pm$ are becoming complex and we obtain corresponding class of classical r -matrices for complex inhomogeneous $\mathfrak{io}(4, C)$ algebra.

iii) Extend supersymmetrically the complex classical r -matrices obtained in ii) to N=1 and N=2 by adding suitable terms which depend on supercharges $Q_\alpha^a, \bar{Q}_{\dot{\alpha}a}$ ($a = 1 \dots N$). For N=2 we consider as well terms in classical r-matrix which depend on complex $N = 2$ central charges Z, \tilde{Z} (see (3.2)) in such a way that the supersymmetric N=2 complex r -matrices satisfy the classical super-YB equation.

In Sect. 4.2 we recall the Zakrzewski list of D=4 real Poincaré and D=4 self-conjugate Euclidean r-matrices from [1] as well as D=4 N=1 real super-Poincaré and pseudoreal super-Euclidean r-matrices obtained in [19]. In Sect. 5 we present new results for supersymmetric r-matrices with standard N=2 Poincaré reality conditions (see Sect. 3.2) and N=2 Euclidean (pseudo)reality conditons (see Sect. 3.3–4).

4.2. D=4 Poincaré real r-matrices

Let us present the real D=4 Poincaré r-matrices listed in [1] (see also [5]). Using the decomposition of $r \in \mathfrak{io}(3, 1) \wedge \mathfrak{io}(3, 1)$

$$r = a + b + c \quad (4.1)$$

where (\mathbb{P} denotes the fourmomenta generators)

$$a \in \mathbb{P} \wedge \mathbb{P} \quad b \in \mathbb{P} \wedge \mathfrak{o}(3, 1) \quad c \in \mathfrak{o}(3, 1) \wedge \mathfrak{o}(3, 1). \quad (4.2)$$

Zakrzewski [1] obtained the following list

where we use Cartan–Chevalley basis for $\mathfrak{o}(3, 1)$ (see (2.17–2.18)) and $P_\pm = P_0 \pm P_3$. Besides b_{P_+}, b_{P_2} are given by the expressions:

$$\begin{aligned} b_{P_+} &= P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h, \\ b_{P_2} &= 2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-, \end{aligned} \quad (4.3)$$

and provide r-matrices describing light-cone (b_{P_+}) and tachyonic (b_{P_2}) κ -deformation [38, 39].

⁷ From the list of Zakrzewski's 21 cases of Poincaré r -matrices, given in [1] only one set, denoted with $\mathcal{N} = 6$, does not satisfy homogeneous classical YB equation, i.e. cannot be lifted to twisted Hopf algebra.

c	b	a	#	\mathcal{N}
$\gamma h' \wedge h$	0	$\alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2$	2	1
$\gamma e'_+ \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	1	2
	$\beta_1 b_{P_+}$	$\alpha P_+ \wedge P_1$	1	3
	$\gamma \beta_1 (P_1 \wedge e_+ + P_2 \wedge e'_+)$	$P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) - \gamma \beta_1^2 P_1 \wedge P_2$	2	4
$\gamma(h \wedge e_+ - h' \wedge e'_+)$ $+ \gamma_1 e'_+ \wedge e_+$	0	0	1	5
$\gamma h \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_2 \wedge e_+$	0	1	6
0	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	1	7
	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+$	0	1	8
	$P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) + \beta_1 P_+ \wedge (h + \sigma e_+), \sigma = 0, \pm 1$	$\alpha P_+ \wedge P_2$	2	9
	$\beta_1 (P_1 \wedge e'_+ + P_+ \wedge e_+)$	$\alpha_1 P_- \wedge P_1 + \alpha_2 P_+ \wedge P_2$	2	10
	$\beta_1 P_2 \wedge e_+$	$\alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2$	1	11
	$\beta_1 P_+ \wedge e_+$	$P_- \wedge (\alpha P_+ + \alpha_1 P_1 + \alpha_2 P_2) + \tilde{\alpha} P_+ \wedge P_2$	3	12
	$\beta_1 P_0 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	2	13
	$\beta_1 P_3 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	2	14
	$\beta_1 P_+ \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	1	15
	$\beta_1 P_1 \wedge h$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	2	16
	$\beta_1 P_+ \wedge h$	$\alpha P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1$	1	17
	$P_+ \wedge (\beta_1 h + \beta_2 h')$	$\alpha_1 P_1 \wedge P_2$	1	18
	0	$\alpha_1 P_1 \wedge P_+$	0	19
		$\alpha_1 P_1 \wedge P_2$	0	20
		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	1	21

Table 1. Real D=4 (pseudo)real Poincaré r-matrices (all satisfy homogeneous CYBE except $\mathcal{N}=6$).

4.3. N=1 D=4 (pseudo) real Poincaré and Euclidean supersymmetric r-matrices

In [19] we have presented the list of possible supersymmetric D=4 N=1 super-Poincaré r-matrices $r^{(1)}$. It appears that only 7 out of 21 classes of r-matrices present in Table 1 can be supersymmetrized. The N=1 super-Poincaré r-matrices decomposed as follows:

$$r^{(1)} = r + s = a + b + c + s, \quad (4.4)$$

where $s \in \mathbb{Q}^{(1)} \wedge \mathbb{Q}^{(1)}$ ($\mathbb{Q}^{(1)}$ denote N=1 Poincaré supercharges). The list of r-matrices (4.4) looks as follows:

c	b	a	s	\mathcal{N}
$\gamma e'_+ \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	$\beta_1 \bar{Q}_1 \wedge Q_1$	2
	$\beta_1 b_{P_+}$	$\alpha P_+ \wedge P_1$	$\beta_1 \bar{Q}_1 \wedge Q_1$	3
$\gamma h \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_2 \wedge e_+$	0	$i\beta_1 (Q_1 + \bar{Q}_1) \wedge (Q_2 - \bar{Q}_2)$	6
0	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	$\beta_1 \bar{Q}_1 \wedge Q_1$	7
	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+$	0	$\beta_1 \bar{Q}_1 \wedge Q_1$	8
	$P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) + \beta_1 P_+ \wedge h$	$\alpha P_+ \wedge P_2$	$\beta_1 \bar{Q}_1 \wedge Q_1$	9
		$\alpha_2 P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1$	$\beta_1 \bar{Q}_1 \wedge Q_1$	17

Table 2. Real D= 4 N=1 super-Poincaré r-matrices (all satisfy homogeneous CYBE except $\mathcal{N}=6$).

We see that the superextension is realized in all cases except $\mathcal{N}=6$ with the help of unique supersymmetric term $S = \beta_1 \bar{Q}_1 \wedge Q_1$, where β_1 is purely imaginary, which is invariant under the N=1 super-Poincaré conjugation.

The list of N=1 complex Euclidean supersymmetric r-matrices, which are self-conjugate under the pseudoconjugation (2.35) with $\eta = -1$ looks as follows:

c	b	a	s	\mathcal{N}
$\gamma h' \wedge h$	0	$\alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2$	$\beta Q_2 \wedge Q_1$	1
$\beta_1 P_0 \wedge h'$		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\beta Q_2 \wedge Q_1$	13
$\beta_1 P_3 \wedge h'$		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\beta Q_2 \wedge Q_1$	14
$\beta_1 P_+ \wedge h'$		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\beta Q_2 \wedge Q_1$	15
$\beta_1 P_1 \wedge h$		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\beta Q_2 \wedge Q_1$	16
	0	$\alpha_1 P_1 \wedge P_+$	$\eta^{\alpha\beta} Q_\alpha \wedge Q_\eta$	19
		$\alpha_2 P_1 \wedge P_2$	$\eta^{\alpha\beta} Q_\alpha \wedge Q_\beta$	20
		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\eta^{\alpha\beta} Q_\alpha \wedge Q_\beta$	21

Table 3. Pseudoreal (selfconjugate under pseudoreality condition) D= 4 N=1 super-Euclidean r-matrices (all satisfy homogeneous CYBE).

where $s \in Q_c^{(1)} \wedge Q_c^{(1)}$ and $Q_c^{(1)}$ denote N=1 complex Euclidean supercharges ($Q_\alpha, \bar{Q}_{\dot{\alpha}}$) (see (2.25–2.27)). Due to inner automorphisms of N=1 Euclidean superalgebra the selfconjugate term can be chosen only as $\beta Q_1 \wedge \bar{Q}_1$, with parameter β purely imaginary.

We observe that among supersymmetric r-matrices in Table 2 the r-matrices with $\mathcal{N}=2,3$ and 7,8 contain term b_{P_+} characterizing light-cone κ -deformation [38], and the super r-matrix $\mathcal{N}=6$ contains term b_{P_2} describing tachyonic κ -deformation [39]. It should be added that the standard “time-like” κ -deformation characterized by the term b_{P_0} , is not present in Tables 1–3. Further we comment that in Euclidean case without supersymmetry (Table 1) and with N=1 supersymmetry (Table 3), the r-matrices characterizing κ -deformations are not present.

5. N=2 extensions with central charges of N=1 super-Poincaré and super-Euclidean r-matrices

The N=2 superextensions of D=4 Poincaré and Euclidean r-matrices can be decomposed in the following way compare with formula (4.3)

$$r^{(2)} = \tilde{a} + \tilde{b} + \tilde{c} + \tilde{s}, \quad (5.1)$$

where $\tilde{a}, \tilde{b}, \tilde{c}$ contains contributions from central charges $\mathbb{Z} = (Z_1, Z_2)$ ($\mathbb{Z} \wedge \mathbb{Z} \equiv Z_1 \wedge Z_2$)

$$\begin{aligned} \tilde{a} &\in \mathbb{P} \wedge \mathbb{P} \\ \tilde{b} &\in \mathbb{P} \wedge (\mathfrak{o}(3,1) \oplus \mathbb{Z}) \\ \tilde{c} &\in \mathfrak{o}(3,1) \oplus \mathbb{Z} \wedge (\mathfrak{o}(3,1) \oplus \mathbb{Z}) \end{aligned} \quad (5.2)$$

and $\tilde{s} \in \mathbb{Q}^{(2)} \wedge \mathbb{Q}^{(2)}$, where $\mathbb{Q}^{(2)}$ denote N=2 supercharges.

We shall list in Sect. 5.1 the r-matrices (5.2) invariant under N=2 Poincaré reality condition (3.10)–(3.11) and in Sect. 5.2 the ones invariant under the pseudoconjugation (3.16) and conjugation (3.24).

5.1. N=2 real super-Poincaré r-matrices

Due to the reality condition (3.11) N=2 Poincaré superalgebra depends on one complex supercharge $\mathbb{Z} \equiv Z_1 = Z_2^\dagger$. In such a case to every Zakrzewski r-matrix (see Table 1) one can add unique term bilinear in central charges (see (5.2))

$$Z \wedge Z^\dagger \in \mathbb{Z} \wedge \mathbb{Z} \in \tilde{c}. \quad (5.3)$$

It follows from Table 2 that only Zakrzewski r-matrices with $\mathcal{N} = 2, 3, 6, 7, 8, 9, 17$ admit $N = 1$ supersymmetrization, realized by universal term $\bar{Q}_1 \wedge Q_1$. It appears that for N=2 D=4 super-Poincaré r-matrices the term \tilde{s} in formula (5.1) is also universal and described by (α real, $\chi, \chi' = 0, \pm 1$)

$$\tilde{s} = \alpha (\chi Q_1^1 \wedge \bar{Q}_1^1 + \chi' Q_1^2 \wedge \bar{Q}_1^2). \quad (5.4)$$

The term (5.4) provides N=2 supersymmetrization of r-matrices⁸ described in Table 2.

If we denote $\tilde{b} = b + \Delta b$ ($\Delta b \in \mathbb{P} \wedge \mathbb{Z}$; see (5.2)) such a term is possible only for supersymmetric r-matrices if $N \geq 2$. These additional terms linearly dependent on central charges, can be added only for $N = 2, 6$ by the following universal expression ($p_+ = p_0 + p_3$; β complex)

$$\Delta b = p_+ \wedge (\beta Z + \bar{\beta} \bar{Z}). \quad (5.5)$$

For $N = 3, 7, 8, 9, 17$ one can add consistently with CYBE the following complex term belonging to $\mathfrak{o}(3, 1) \wedge \mathbb{Z} \in \tilde{c}$ (β_1, β_2 – complex)

$$\Delta c = \beta_1 e_+ \wedge Z + \beta_2 e'_+ \wedge \bar{Z}. \quad (5.6)$$

Unfortunately, term (5.6) can not satisfy the Poincaré reality conditions.

Listed above supersymmetric r -matrices can be presented as a sum of subordinated r -matrices which are of super-Abelian and super-Jordanian types. The subordination enables us to construct a correct sequence of quantizations and to obtain the corresponding twists describing the quantum deformations. These twists are in general case the super-extensions of the twists obtained in [5].

5.2. N=2 (pseudo) real super-Euclidean r-matrices

Contrary to the N=2 super-Poincaré case, when there is only one conjugation providing reality condition (see Sect. 3.2⁹), in N=2 case we have two types of reality structure:

- a) Defined by pseudoconjugation (3.16) (we consider $q = 0$ and $\eta = -1$), which is the straightforward extension of pseudoreality structure considered for N=1 in (2.35).
- b) Defined by the conjugation (3.24). We further assume that the Hermitean conjugation $Q_\alpha^a \rightarrow (Q_\alpha^a)^*$, $\bar{Q}_\alpha^a \rightarrow (\bar{Q}_\alpha^a)^*$ is well defined.¹⁰ Then one can formulate N=2 Euclidean superalgebra in a Hermitean form if we impose the subsidiary condition which follows from (3.24) ($\alpha = 1, 2$; $a, b = 1, 2$)

$$Q_\alpha^a = (Q_\alpha^a)^\dagger \Rightarrow Q_\alpha^a = \varepsilon_{\alpha\beta} \varepsilon^{ab} (\bar{Q}_\beta^b)^* \quad \bar{Q}_\alpha^a = (\bar{Q}_\alpha^a)^\dagger \Rightarrow \bar{Q}_{\dot{\alpha}a} = \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{ab} Q_{\dot{\beta}b}, \quad (5.7)$$

or more explicitly

$$\begin{aligned} Q_\alpha^1 &= \varepsilon_{\alpha\beta} (\bar{Q}_\beta^2)^* & Q_\alpha^2 &= -\varepsilon_{\alpha\beta} (\bar{Q}_\beta^1)^*, \\ \bar{Q}_{\dot{\alpha}}^1 &= \varepsilon_{\dot{\alpha}\dot{\beta}} (Q_{\dot{\beta}}^2)^* & \bar{Q}_{\dot{\alpha}}^2 &= -\varepsilon_{\dot{\alpha}\dot{\beta}} (Q_{\dot{\beta}}^1)^*. \end{aligned} \quad (5.8)$$

We see that the supercharges $Q_\alpha^2, \bar{Q}_{\dot{\alpha}}^2$ can be expressed by $(\bar{Q}_\alpha^1)^*, (Q_\alpha^1)^*$ and N=2 superalgebra can be described by two pairs of complex Hermitean-conjugated supercharges which equivalently can be expressed as Hermitean-conjugated pair of four-component Dirac spinors (A=1,2,3,4)

$$\Psi_A = (Q_\alpha^1, \bar{Q}_{\dot{\alpha}}^1), \quad \Psi_A^* = ((Q_\alpha^1)^*, (\bar{Q}_{\dot{\alpha}}^1)^*). \quad (5.9)$$

We shall consider below separately the N=2 Euclidean r-matrices selfconjugate under pseudoconjugation (3.16) and conjugation (5.7).

⁸ If χ or $\chi' = 0$ then we obtain only N=1 supersymmetrization.

⁹ We considered only here the reality structure defined by standard antilinear antiinvolution ($q = 0$ in relation (A.1)) what corresponds to the choice $\eta = 1$ in (3.10).

¹⁰ Such situation occurs in supersymmetric QFT, with supercharges realized as differential operators on superspace fields. In fact the conjugation (3.24) can be defined for suitable class of realizations of supercharges $Q_\alpha^a, \bar{Q}_{\dot{\alpha}}^a$.

5.2.1. N=2 super-Euclidean r-matrices selfconjugate under pseudoconjugation We consider the complex N=2 $\varepsilon(4;2|\mathbb{C})$ r-matrices which are invariant under the map (3.16); we choose the standard version of formula (A.1) with $q = 0$ what implies $\eta = -1$. For N=2 we should take into consideration only the Poincaré r-matrices from Table 1 with $\mathcal{N} = 1, 13 - 16, 19 - 21$ (see also Table 3) which allow N=1 supersymmetrization. If we consider the relations (3.17) with $q = 0$, we get the pair of independent N=2 Euclidean central charges Z, \tilde{Z} which are purely imaginary. They provide Euclidean counterpart of formula (5.3) describing universal contribution to N=2 super-r-matrices.

The fermionic part \tilde{s}_E of N=2 super-Euclidean r-matrix (5.1) for $\mathcal{N} = 1, 13 - 16$ is described by the following pair of two forms bilinear in supercharges ($\alpha_1, \alpha_2, \tilde{\alpha}_1, \tilde{\alpha}_2$ are real)

$$\begin{aligned}\tilde{s}_{E;1} &= \alpha_1 Q_1^1 \wedge Q_2^1 + \alpha_2 \bar{Q}_1^2 \wedge \bar{Q}_2^2, \\ \tilde{s}_{E;2} &= \tilde{\alpha}_1 Q_1^2 \wedge Q_2^2 + \tilde{\alpha}_2 \bar{Q}_1^1 \wedge \bar{Q}_2^1.\end{aligned}\tag{5.10}$$

To either of two terms (5.10) one can add the unique term $\Delta c_E \in \mathfrak{o}(3,1) \wedge \mathbb{Z}$ which takes the form (α complex, β_1, β_2 real)

$$\Delta c_E = (\alpha h + \bar{\alpha} h')(\beta_1 Z + \beta_2 \tilde{Z}).\tag{5.11}$$

For $\mathcal{N} = 19 - 21$ we can again choose pair of the purely fermionic terms \tilde{s}_E , which are described by formulae (5.10); the terms Δc_E , linear in Z, \tilde{Z} , are however not universal, different for three cases $\mathcal{N} = 19, 20$ and 21 .

5.2.2. N=2 real super-Euclidean r-matrices In this case the algebraic structure does not have a counterpart in the formulae obtained by Euclidean N=1 supersymmetrization (see Table 3). The task consists in finding such N=2 complex $\varepsilon(4;2|\mathbb{C})$ r-matrices which are consistent with N=2 super-Euclidean reality conditions (5.8). We should mention that it is necessary to consider the N=2 supersymmetrization of Poincaré r-matrices for all $\mathcal{N} = 1 \dots 21$. We obtain the following list of fermionic and central charge dependent terms for various choices of \mathcal{N}

$\alpha)$ $\mathcal{N} = 1, 13 - 16, 18$

There are possible the following independent four fermionic two-forms $\tilde{s}_{E;k}$ ($k = 1, 2, 3, 4$; $\alpha_1, \dots, \alpha_6$ real)

$$\begin{aligned}\tilde{s}_{E;1} &= \alpha_1 (Q_1^1 \wedge Q_2^1 + \bar{Q}_1^2 \wedge \bar{Q}_2^2), \\ \tilde{s}_{E;2} &= \alpha_2 (Q_1^2 \wedge Q_2^2 + \bar{Q}_1^1 \wedge \bar{Q}_2^1), \\ \tilde{s}_{E;3} &= i\alpha_3 Q_1^1 \wedge \bar{Q}_2^2 + i\alpha_4 Q_2^1 \wedge \bar{Q}_1^2, \\ \tilde{s}_{E;4} &= i\alpha_5 Q_1^2 \wedge \bar{Q}_2^1 + i\alpha_6 Q_2^2 \wedge \bar{Q}_1^1.\end{aligned}\tag{5.12}$$

The additional bosonic terms $\Delta c_E \in \mathfrak{o}(3,1) \wedge \mathbb{Z}$ which are linear in real central charges Z, \tilde{Z} satisfying (for $\eta = 1$) the reality conditions (3.21) is described again by formula (5.11).

$\beta)$ $\mathcal{N} = 19 - 21$.

For such values of \mathcal{N} one can add any of four three-parameter term $\tilde{\tilde{s}}_{E;k}$ below, bilinear in supercharges ($\alpha_1, \dots, \alpha_{12}$ real):

$$\begin{aligned}
\tilde{s}_{E;1} &= \alpha_1(Q_1^1 \wedge Q_1^1 + \bar{Q}_2^2 \wedge \bar{Q}_2^2) + \alpha_2(Q_1^1 \wedge Q_2^1 + \bar{Q}_1^2 \wedge Q_2^2) + \alpha_3(Q_2^1 \wedge Q_2^1 + \bar{Q}_1^2 \wedge \bar{Q}_1^2), \\
\tilde{s}_{E;2} &= \alpha_4(Q_2^1 \wedge Q_2^1 + Q_1^2 \wedge \bar{Q}_1^2) + \alpha_5(Q_1^2 \wedge Q_2^2 + \bar{Q}_1^1 \wedge \bar{Q}_2^1) + \alpha_6(Q_2^2 \wedge Q_2^2 + \bar{Q}_1^1 \wedge \bar{Q}_1^1), \\
\tilde{s}_{E;3} &= \alpha_7 Q_1^1 \wedge \bar{Q}_2^2 + \alpha_8 Q_2^1 \wedge \bar{Q}_1^2 + \alpha_9(Q_1^1 \wedge \bar{Q}_1^2 + Q_2^1 \wedge \bar{Q}_2^2), \\
\tilde{s}_{E;4} &= \alpha_{10} Q_1^2 \wedge \bar{Q}_2^1 + \alpha_{11} Q_2^2 \wedge \bar{Q}_1^1 + \alpha_{12}(Q_1^2 \wedge \bar{Q}_1^1 + Q_2^2 \wedge \bar{Q}_2^1).
\end{aligned} \tag{5.13}$$

The additional bosonic terms Δb_E linear in central charges and fourmomenta are different for considered three classes of r-matrices, namely $(\beta_1, \beta_2, \gamma_2, \gamma_3, \gamma_0)$ real

$$\begin{aligned}
N=19: \quad \Delta b_E &= p_1 \wedge (\beta_1 Z + \beta_2 \tilde{Z}), \\
N=20: \quad \Delta b_E &= (p_1 + \gamma_2 p_2) \wedge (\beta_1 Z + \beta_2 \tilde{Z}), \\
N=21: \quad \Delta b_E &= (p_1 + \gamma_2 p_2 + \gamma_3 p_3 + i\gamma_0 p_0) \wedge (\beta_1 Z + \beta_2 \tilde{Z}).
\end{aligned} \tag{5.14}$$

Finally we point out that among all 21 Zakrzewski classes of complex $\mathfrak{io}(4; \mathbb{C})$ r-matrices only for $\mathcal{N}=5$ we could not find the superextension of complex $\mathfrak{o}(4; \mathbb{C})$ r-matrices to $\mathfrak{e}(4; N|\mathbb{C})$ ($N=1, 2$) supersymmetric r-matrices, i.e. we were not able for $\mathcal{N}=5$ to provide any consistent fermionic term \tilde{s} in (5.1).

6. Final remarks

This paper provides firstly systematic discussion of real forms of $\mathfrak{o}(4; \mathbb{C})$, $\mathfrak{io}(4; \mathbb{C})$, $\mathfrak{e}(4; 1|\mathbb{C})$ and $\mathfrak{e}(4; 2|\mathbb{C})$, where $\mathfrak{e}(4; N|\mathbb{C})$ describes complex D=4 N-extended Euclidean superalgebra. In Sect. 2 and 3 we consider the reality and pseudoreality conditions (reality constraints). To the Poincaré and Euclidean (pseudo)real forms we added also the (pseudo)real forms for the Kleinian signature $g_{\mu\nu} = \text{diag}(1, -1, 1, -1)$. In particular we also considered the (pseudo) reality conditions leading to exotic supersymmetry scheme, with odd (Grassmann) coordinates conjugated in nonstandard way (see e.g. (2.38)–(2.39), with $q=1$).

Our second aim was to present the extension of Zakrzewski list of classical D=4 Poincaré r-matrices to Euclidean case and N=1,2 supersymmetrizations. The D=4 N=1 Poincaré and Euclidean supersymmetric classical r-matrices already considered in [19] were presented in Sect. 4. In Sect. 5 we describe partial results for the classification of D=4 N=2 supersymmetric r-matrices for various D=4 signatures. We consider in this paper N=2 Poincaré and Euclidean signatures and these results are new. It should be pointed out that we considered also the terms depending on a pair of N=2 central charges (for $\mathfrak{e}(4; 2|\mathbb{C})$); they are complex-conjugated for Poincaré and real for Euclidean cases.

We did not consider the deformations of N=2 Kleinian supersymmetry and corresponding classical r-matrices. More systematic approach, with more complete list of complex classical $\mathfrak{e}(4; 2|\mathbb{C})$ r-matrices and their various real forms will be considered in our subsequent publication.

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Appendix A. Conjugations and pseudoconjugations of complex Lie superalgebras

In order to describe the supersymmetries of physical systems one should consider real and pseudoreal forms of complex Lie superalgebra L , which are defined in algebraic framework with help of the

conjugations and pseudoconjugations. The conjugations and pseudoconjugations are usually defined as abstract antilinear antiautomorphisms $x \rightarrow x^*$ of second and fourth order preserving the Z_2 grading of superalgebra and satisfying the properties $(x, y \in U_L; |x| = 0 \ (|x| = 1)$ describes the parity of even (odd) element x), where U_L denotes enveloping algebra of L :

$$(xy)^* = (-1)^{q|x||y|} y^* x^* \quad q = 0, 1 \quad (\text{A.1})$$

$$(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^* \quad \alpha, \beta \in C \quad (\text{A.2})$$

where $q = 0, 1$ defines two types of antilinear antiinvolution map in U_L and $\alpha \rightarrow \bar{\alpha}, \beta \rightarrow \bar{\beta}$ describe complex conjugation in C . Further

$$(x^*)^* = x \quad (\text{conjugation}), \quad (\text{A.3a})$$

$$(x^*)^* = -x \quad (\text{pseudoconjugation}). \quad (\text{A.3b})$$

For Lie superalgebras the property (A.3b) occurs only in odd parity sector, i.e. for any $x \in L$ both relations (A.3a) and (A.3b) can be written together as

$$(x^*)^* = (-1)^{p|x|} x, \quad (\text{A.3c})$$

where $p = 0$ (resp. $p = 1$) denotes conjugations (resp. pseudoconjugation) and we recall that $|x|$ describe the grading of superalgebra element x . The conjugations (A.3a) in matrix and Hilbert space realizations of superalgebra can be identified with the Hermitean conjugation, and pseudoconjugations in odd sector of the matrix superalgebras with $p = 1$ were introduced as graded Hermitean conjugation [26] (see also [27]). In the case of conjugations (A.3a) the Hermitean elements L_R, \tilde{L}_R of complex Lie algebra L are defined as follows

$$L_R : x_R = (x + x^*) \quad \tilde{L}_R : \tilde{x}_R = i(x - x^*) \quad x \in L, \quad (\text{A.4})$$

where $x_R^+ = -x_R$ and $\tilde{x}_R^+ = -\tilde{x}_R$. In the case of conjugation the superalgebras L_R, \tilde{L}_R are the subsuperalgebras which can be defined as fixed points of the conjugation map, and $L = L_R \oplus \tilde{L}_R$ provides the formula describing the realification of L . In the case of pseudoconjugations (A.3b) the elements (A.4) of complex Lie algebra satisfy the set of relations

$$x_R^\# = -(x - x^\#) = i\tilde{x}_R \quad \tilde{x}_R^\# = -ix_R \quad (\text{A.5})$$

with elements x_R, \tilde{x}_R satisfying involutive relations of fourth order

$$(x_R^\#)^\# = -x_R \quad (x_L^\#)^\# = -x_L. \quad (\text{A.6})$$

We comment that one uses for description of real supersymmetries in classical physics an alternative conjugations and pseudoconjugations, which are antilinear automorphism with the property

$$\overline{\bar{x}y} = \bar{x}\bar{y} \quad ; \quad (\text{A.7})$$

the remaining relations (A.2), (A.3a–b) are unchanged. One can add that automorphisms (A.7) for classical physical systems are usually represented by complex conjugation, and antiautomorphisms adjusted to quantum systems are realized as (graded) Hermitean conjugation in suitable Hilbert space framework.

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